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AN EFFICIENT SOLUTION OF PRANDTL-TYPE INTEGRODIFFERENTIAL EQUATIONS IN A SECTION AND ITS APPLICATION TO CONTACT PROBLEMS FOR A STRIP†

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Two Prandtl-type integrodifferential equations are solved exactly, one equation arising from the antiplane problem for an elastic layer, one of whose boundaries is rigidly attached, the other boundary being rigidly attached everywhere except along a section where it is elastically attached, the other equation arising from the plane problem of a strip-shaped membrane uniformly extending at infinity and strengthened by elastic inclusions. In both cases the integral equation leads, with the help of a Fourier transformation, to a vector Riemann problem, which reduces by a method similar to one presented earlier [1] to an infinite Poincaré–Koch algebraic system. Explicit formulae are found for the system unknowns together with recurrence relations that are convenient for numerical implementation. Computational formulae are found for the axial forces at the ends of the stringer, together with tangential contact stresses and their intensity factors. In the neighbourhood of the ends of the stringer an asymptotic expansion for the contact stresses is constructed, which, besides powers of radicals, contains products of radicals in integer powers of logarithms. Numerical results are presented.

AN INTEGRODIFFERENTIAL equation arising from the plane problem of the extension of a strip with covering was solved approximately in [2] using an asymptotic method and the method of successive approximations. The equation of the Prandtl problem for the contact of a half-plane with an adhesive covering was reduced [3, 4] to an infinite algebraic system with a power-law decrease of the elements of the system matrix. This equation was solved [2] by an asymptotic method.

1. THE PRANDTL-TYPE EQUATION FOR AN ANTIPLANE STRIP PROBLEM

Consider the following harmonic problem for a strip

$$\Delta w(x,y)=0, \quad |x|<\infty, \quad 0<y<b \tag{1.1}$$

$$w(x,0)=0, \quad x \overline{\in} (0,a); \quad w(x,b)=0; \ |x|<\infty \tag{1.2}$$

$$(w-\mu_0 \partial w/\partial y)_{y=0}=f_0(x), \quad 0<x<a \tag{1.3}$$

Here $\mu_0>0$ and $f_0(x)$ is Hölder's function.

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We extend the first condition of (1.2) over the entire real axis

$$w(x,0)=\chi_0(x),\quad |x|<\infty,\quad \operatorname{supp}\chi_0(x)\subset(0,a)$$

and apply a Fourier transformation to problem (1.1), (1.2). The implementation of condition (1.3) leads to the Prandtl integrodifferential equation

$$\chi(t)+\frac{\mu}{2}\int_0^\lambda \chi'(\tau)\operatorname{cth}\frac{\pi}{2}(t-\tau)\,d\tau=f(t),\quad 0<t<\lambda \tag{1.4}$$

with the additional condition $\chi(0)=\chi(\lambda)=0$, where

$$\chi(t)=\chi_0(bt),\quad f(t)=f_0(bt),\quad \lambda=a/b,\quad \mu=\mu_0/b$$

With the help of the single-sided functions $\chi_\pm(t)$ and the functions $\chi_*(t)$ and $f_*(t)$ possessing the properties

$$\operatorname{supp}\chi_+\subset[\lambda,\infty),\quad \operatorname{supp}\chi_-\subset(-\infty,0]$$

$$\chi_*(t)=\begin{cases}\chi(t), & 0\leqslant t\leqslant\lambda\\ 0, & t\,\overline{\in}\,(0,\lambda)\end{cases},\qquad f_*(t)=\begin{cases}f(t), & 0\leqslant t\leqslant\lambda\\ 0, & t\,\overline{\in}\,(0,\lambda)\end{cases}$$

we extend Eq. (1.4) over the entire axis

$$\chi_*(t)+\frac{\mu}{2}\int_{-\infty}^{\infty}\chi_*'(\tau)\operatorname{cth}\frac{\pi}{2}(t-\tau)\,d\tau = f_*(t)+\chi_-(t)+\chi_+(t),\quad |t|<\infty \tag{1.5}$$

We introduce the Fourier transforms

$$\Phi_1^+(\alpha)=\int_0^\lambda \chi(\tau)e^{i\alpha\tau}d\tau,\quad \Phi_1^-(\alpha)=\int_{-\lambda}^0\chi(\lambda+\tau)e^{i\alpha\tau}d\tau$$

$$\Phi_2^+(\alpha)=\int_0^\infty \chi_+(\lambda+\tau)e^{i\alpha\tau}d\tau,\quad \Phi_2^-(\alpha)=\int_{-\infty}^0\chi_-(\tau)e^{i\alpha\tau}d\tau$$

$$F^+(\alpha)=\int_0^\lambda f(t)e^{i\alpha t}dt,\quad F^-(\alpha)=\int_{-\lambda}^0 f(\lambda+t)e^{i\alpha t}dt$$

The functions $\Phi_1^\pm(\alpha)$ and $F^\pm(\alpha)$ are entire, and $\Phi_2^\pm(\alpha)$ are analytic in $\mathbf{C}^\pm$: $\operatorname{Im}\alpha\gtrless 0$. Applying a Fourier transformation to Eq. (1.5) and using the relation

$$\Phi_1^+(\alpha)=e^{i\alpha\lambda}\Phi_1^-(\alpha)$$

we obtain the following vector Riemann problem

$$G(\alpha)\Phi_1^\pm(\alpha)=F^\pm(\alpha)+e^{\pm i\alpha\lambda}\Phi_2^\pm(\alpha)+\Phi_2^\mp(\alpha),\quad |\alpha|<\infty \tag{1.6}$$

$$G(\alpha)=1+\mu\alpha\operatorname{cth}\alpha$$

We factorize the function $G(\alpha)$

$$\begin{gathered}G(\alpha)=K^+(\alpha)X^+(\alpha)K^-(\alpha)X^-(\alpha)\\ K^\pm(\alpha)=(\pi\mu)^{1/2}\Gamma(1\mp i\alpha/\pi)[\Gamma(1/2\mp i\alpha/\pi)]^{-1}\\ X^\pm(\alpha)=X^{\pm1}(\alpha),\quad \alpha\in\mathbf{C}^\pm,\quad X(\alpha)=\exp\left(\frac{\alpha}{\pi i}\int_0^\infty \ln G_0(x)\frac{dx}{x^2-\alpha^2}\right)\end{gathered} \tag{1.7}$$

$$G_0(\alpha)=1+(\mu\alpha)^{-1}\operatorname{th}\alpha, \quad \operatorname*{ind}_{(-\infty,+\infty)} G_0(\alpha)=0$$

and solve the jump problem

$$\omega_\pm^+(\alpha)-\omega_\pm^-(\alpha) = \frac{F^\pm(\alpha)}{K^\mp(\alpha)X^\mp(\alpha)}, \quad \omega_\pm(\alpha)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{F^\pm(x)\,dx}{K^\mp(x)X^\mp(x)(x-\alpha)} \tag{1.8}$$

We rewrite boundary condition (1.6) in the form

$$K^\pm(\alpha)X^\pm(\alpha)\Phi_1^\pm(\alpha)-e^{\pm i\alpha\lambda}[G(\alpha)]^{-1}K^\pm(\alpha)X^\pm(\alpha)\Phi_2^\pm(\alpha)\mp\omega_\pm^\pm(\alpha)=$$
$$=[K^\mp(\alpha)X^\mp(\alpha)]^{-1}\Phi_2^\mp(\alpha)\mp\omega_\pm^\mp(\alpha), \quad |\alpha|<\infty \tag{1.9}$$

The function $G(\alpha)=1+\mu\alpha\operatorname{cth}\alpha$ has a denumerable set of zeros $\alpha_n=\pm i\beta_n\in\mathbf{C}^\pm$ $(n=1, 2, \ldots)$, where all the β_n are real and have the asymptotic form $\beta_n=\pi(n-\tfrac{1}{2})+o(1)$, $n\to\infty$. To remove the poles of the function $[G(\alpha)]^{-1}$ we remove from the left- and right-hand sides of (1.9) the function

$$\Psi^\pm(\alpha)=\sum_{n=1}^{\infty}\frac{iA_n^\pm}{\alpha\pm i\beta_n} \tag{1.10}$$

and require that the conditions

$$\operatorname*{res}_{\alpha=\pm i\beta_n}\{-e^{\pm i\alpha\lambda}[G(\alpha)]^{-1}K^\pm(\alpha)X^\pm(\alpha)\Phi_2^\pm(\alpha)-\Psi^\mp(\alpha)\}=0 \quad (n=1,2,\ldots) \tag{1.11}$$

be satisfied.

Subsequent use of Liouville's theorem leads to formulae giving the solution of the Riemann problem (1.6)

$$\Phi_1^-(\alpha) = \frac{e^{-i\alpha\lambda}}{G(\alpha)}K^-(\alpha)X^-(\alpha)[\omega_+^-(\alpha)+\Psi^-(\alpha)]+\frac{\Psi^+(\alpha)-\omega_-^-(\alpha)}{K^-(\alpha)X^-(\alpha)}$$

$$\Phi_1^+(\alpha)=e^{i\alpha\lambda}\Phi_1^-(\alpha), \quad \Phi_2^\pm(\alpha)=K^\pm(\alpha)X^\pm(\alpha)[\Psi^\pm(\alpha)\mp\omega_\mp^\pm(\alpha)] \tag{1.12}$$

Substituting formulae (1.12) into conditions (1.11), we arrive at an infinite Poincaré–Koch linear algebraic system

$$A_n^\pm=e^{-\lambda\beta_n}\Delta_n\left(f_n^\mp+\sum_{m=1}^{\infty}\frac{A_m^\mp}{\beta_n+\beta_m}\right) \tag{1.13}$$

$$f_n^\pm=-\omega_\mp^\pm(\pm i\beta_n), \quad \Delta_n=K_n^2X_n^2G_n^{-1}, \quad G_n=\mu\beta_n-(\mu+1)\operatorname{ctg}\beta_n$$
$$K_n=\frac{(\pi\mu)^{1/2}\Gamma(1+\pi^{-1}\beta_n)}{\Gamma(\tfrac{1}{2}+\pi^{-1}\beta_n)}, \quad X_n=\exp\left(\frac{\beta_n}{\pi}\int_0^{\infty}\ln G_0(x)\frac{dx}{x^2+\beta_n^2}\right) \tag{1.14}$$

the solution of which is given by the recurrence relations

$$A_n^\pm=e^{-\lambda\beta_n}\sum_{k=0}^{\infty}a_{nk}^\pm, \quad a_{n0}^\pm=\Delta_n f_n^\mp, \quad a_{np}^\pm=\Delta_n\sum_{j=1}^{p}\frac{a_{j,p-j}^\mp}{\beta_n+\beta_j}e^{-\lambda\beta_j} \tag{1.15}$$

From this we obtain

$$A_n^\pm=O(n^{-1}e^{-\lambda\beta_n}), \quad n\to\infty$$

which ensures that the series (1.10) converges in the $\mathbf{C}^{\pm}$ half-planes.

With the help of an inverse Fourier transformation we find the solution of Eq. (1.4)

$$\chi(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_1^+(\alpha)e^{-i\alpha t}d\alpha$$

Substituting expression (1.12) for Φ_1^+ into this last expression, applying the theory of residues and using relations (1.18) and (1.13), we finally obtain

$$\chi(t)=\sum_{n=1}^{\infty}\frac{e^{\beta_n t}A_n^- - e^{(\lambda-t)\beta_n}A_n^+}{K_n X_n}+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F^+(\alpha)e^{-i\alpha t}}{1+\mu\alpha\,\mathrm{cth}\,\alpha}d\alpha \tag{1.16}$$

We obtain an explicit solution for system (1.13) for the case $f(x)=1$. Then

$$F^+(\alpha)=(e^{i\alpha\lambda}-1)(i\alpha)^{-1}$$

and it is not necessary to solve the jump problem (1.8). Formulae (1.12) acquire the form

$$\begin{aligned}
&\Phi_1^+(\alpha)=\frac{ig_1\alpha^{-1}+\Psi^-(\alpha)}{K^+(\alpha)X^+(\alpha)}+\frac{e^{i\alpha\lambda}}{G(\alpha)}K^+(\alpha)X^+(\alpha)\left[\frac{g_1}{i\alpha}+\Psi^+(\alpha)\right]\\
&\Phi_2^\pm(\alpha)=\pm i\alpha^{-1}+K^\pm(\alpha)X^\pm(\alpha)[\mp ig_1\alpha^{-1}+\Psi^\pm(\alpha)]\\
&g_1=[K^\pm(0)X^\pm(0)]^{-1}=(\mu+1)^{\frac{1}{2}}
\end{aligned} \tag{1.17}$$

The coefficients $f_n^\pm$ are given explicitly, without quadratures $f_n^\pm=\mp\beta_n^{-1}g_1$. As a result of this relations (1.15) simplify

$$\begin{aligned}
&\pm A_n^\pm=A_n,\quad \pm a_{nk}^\pm=a_{nk},\quad a_{n0}=\Delta_n g_1\beta_n^{-1}\\
&A_n=e^{-\lambda\beta_n}\sum_{k=0}^{\infty}a_{nk},\quad a_{np}=-\Delta_n\sum_{j=1}^{p}\frac{e^{-\lambda\beta_j}a_{j,p-j}}{\beta_n+\beta_j}
\end{aligned} \tag{1.18}$$

From this we find an explicit expression for the coefficients a_{np}

$$\begin{aligned}
a_{np}=\Delta_n g_1&\left[-\frac{\Delta_p}{(\beta_n+\beta_p)\beta_p}+\frac{2}{\beta_n+\beta_1}\left(-\frac{e^{-\lambda\beta_1}\Delta_1}{2\beta_1}\right)^p+\right.\\
&\left.+\sum_{m=1}^{p-2}(-1)^{m+1}\sum_{j_1=1}^{\sigma(0)-1}h_1\sum_{j_2=1}^{\sigma(1)-1}h_2\dots\sum_{j_m=1}^{\sigma(m-1)-1}\frac{h_m\Delta_{\sigma(m)}}{\beta_{\sigma(m)}(\beta_{j_m}+\beta_{\sigma(m)})}\right]\\
&\sigma(m)=p-j_1-\dots-j_m\quad(m=1,2,\dots),\quad\sigma(0)=p,\quad j_0=n,\\
&h_m=(\beta_{j_{m-1}}+\beta_{j_m})^{-1}\Delta_{j_m}e^{-\lambda\beta_{j_m}}
\end{aligned} \tag{1.19}$$

In the case under consideration the quadrature in (1.16) does not have to be calculated, and the solution of Eq. (1.4) for $f(t)=1$ has the form

$$\chi(t)=-2\sum_{n=1}^{\infty}\frac{A_n}{K_n X_n}e^{\frac{1}{2}\lambda\beta_n}\,\mathrm{ch}\beta_n\left(t-\frac{\lambda}{2}\right) \tag{1.20}$$

2. EXTENSION OF AN INFINITE ELASTIC STRIP ALONG A STRINGER

Suppose that an elastic strip Π $\{|x|<\infty,\ |y|<b\}$ with modulus of elasticity E and Poisson's ratio ν is reinforced by a stringer $S=\{|x|<a,\ |y|<\frac{1}{2}h\}$ (a thin elastic rod with no bending stiffness) with modulus of elasticity E_0, and is stretched at infinity by uniformly distributed forces of strength q. In a plane stressed state it is required to determine the tangential contact stresses (the normal ones vanishing) and the axial forces at the ends of the inclusion.

From the stringer equilibrium equation we obtain an expression for its horizontal deformations

$$\varepsilon_x^{(0)}(x)=\frac{1}{E_0 h}\left[P_1-\int_{-a}^{x}\tau_0(\xi)\,d\xi\right],\quad |x|<a$$

The equilibrium condition for the inclusion has the form

$$\int_{-a}^{a}\tau_0(x)\,dx=P_1-P_2,\quad \tau_0(x)=\tau_+(x)-\tau_-(x) \tag{2.1}$$

where $\tau_\pm(x)$ are the unknown tangential contact stresses at the upper and lower sides of the inclusion, and P_1 and P_2 are unknown axial forces at its ends $x=-a$ and $x=a$, respectively. Because of the symmetry of the problem, $P_1=P_2=P$, $\tau_+(x)=-\tau_-(x)=\frac{1}{2}\tau_0(x)$.

We will apply a model [5] for the contact between a string and a strip, according to which the horizontal deformations $\varepsilon_x^{(0)}$ and $\varepsilon_x(x,\ 0)$ of the stringer and the homogeneous elastic strip, respectively, are equal, the strip being loaded over the interval $(-x,\ x)$ of the x axis by shear stresses $\tau_0(x)$ and also by forces at infinity. These strip deformations have the form

$$\varepsilon_x(x,0)=\frac{q}{E}+\frac{\partial u}{\partial x}(x,0),\quad E\frac{\partial u}{\partial x}=\frac{\partial^2 U}{\partial y^2}-\nu\frac{\partial^2 U}{\partial x^2}$$

where $U(x,y)$ is a stress function satisfying the following boundary-value problem

$$\begin{gathered}\Delta^2 U(x,y)=0,\quad |x|<\infty,\quad 0<y<b\\ \left.\frac{\partial^2 U}{\partial x^2}\right|_{y=b}=0,\quad -\left.\frac{\partial^2 U}{\partial x\,\partial y}\right|_{y=b}=0,\quad |x|<\infty\\ -\left.\frac{\partial^2 U}{\partial x\,\partial y}\right|_{y=0}=\tau_+(x),\quad \left[\frac{\partial^3 U}{\partial y^3}+(2+\nu)\frac{\partial^3 U}{\partial x^2\,\partial y}\right]_{y=0}=0,\quad |x|<\infty\end{gathered} \tag{2.2}$$

the solution of which is constructed by means of Fourier transformations. We have

$$\left.\frac{\partial u}{\partial x}\right|_{y=0}=\frac{\kappa_0}{\pi E}\int_{-a}^{a}\tau_+(\xi)\int_0^\infty \sin\alpha(x-\xi)\frac{\operatorname{ch}^2\alpha b+\kappa_1\alpha^2 b^2+\kappa_2}{\operatorname{sh}2\alpha b+2\alpha b}\,d\alpha\,d\xi$$

$$\kappa_0=(3-\nu)(1+\nu),\quad \kappa_1=(1+\nu)(3-\nu)^{-1},\quad \kappa_2=(1-\nu)^2\kappa_0^{-1}$$

Bearing in mind the contact condition $\varepsilon_x^{(0)}(x)=\varepsilon_x(x,\ 0)$, $|x|<a$ and introducing a new unknown function

$$\chi(t)=\frac{2b}{P_*}\int_0^t\tau_+(-a+b\xi)\,d\xi,\quad P_*=P-\frac{E_0}{E}qh \tag{2.3}$$

we arrive at the integrodifferential equation

$$\chi(t)+\mu\int_0^\lambda S(t-\tau)\chi'(\tau)d\tau=1, \quad 0<t<\lambda$$

$$\mu=\frac{E_0 h\kappa_0}{4bE}, \quad \lambda=\frac{2a}{b}, \quad S(t)=\frac{1}{\pi}\int_0^\infty \frac{\operatorname{ch}^2\alpha+\kappa_1\alpha^2+\kappa_2}{\operatorname{sh}\alpha\operatorname{ch}\alpha+\alpha}\sin\alpha t\, d\alpha \tag{2.4}$$

with additional conditions

$$\chi(0) = \chi(\lambda) = 0 \tag{2.5}$$

which follow from equalities (2.3) and (2.1) ($P_1=P_2$).

Following the scheme of Sec. 1 (the case when $f(x)=1$), Eq. (2.4) reduces to the matrix Riemann problem (1.6) where

$$G(\alpha)=1+\mu\alpha\frac{\operatorname{ch}^2\alpha+\kappa_1\alpha^2+\kappa_2}{\operatorname{sh}\alpha\operatorname{ch}\alpha+\alpha}, \quad F^{\pm}(\alpha)=\pm\frac{e^{\pm i\alpha}-1}{i\alpha}$$

The factorization of the function $G(\alpha)$ is governed by formulae (1.7), where one must take $G_0(\alpha)$ to be

$$G_0(\alpha)=1+\frac{\operatorname{th}\alpha}{\mu\alpha}+\frac{(\kappa_1\alpha^2+\kappa_2)\operatorname{th}\alpha-\alpha}{\operatorname{sh}\alpha\operatorname{ch}\alpha+\alpha}$$

We will investigate the transcendental equation $G(\alpha)=0$. The functional $G(\alpha)$ has no real roots, and on the imaginary axis there are two symmetrically positioned roots $\pm i\beta_1$ such that when $\mu\to\infty$, $\beta_1\to 0$, whereas when $\mu\to 0$, $\beta_1\to\infty$. Furthermore, the function $G(\alpha)$ has a denumerable set of complex roots $\pm\alpha_m\in\mathbf{C}^{\pm}$, $\alpha_m=i\beta_m$, $\beta_{2m+j}=\frac{1}{2}[b_m-(-1)^j ia_m]$ ($m=1, 2, \ldots$; $j=0, 1$). The numbers $z_m=a_m+ib_m$ are roots of the equation

$$\operatorname{sh} z+z+\mu z(\tfrac{1}{2}\operatorname{ch} z+\tfrac{1}{2}+\tfrac{1}{4}\kappa_1 z^2+\kappa_2)=0$$

and are computed using the iterative formula

$$z_n^{(k)}=2\pi ni+\ln\varphi(z_n^{(k-1)}) \quad (k=2,3,\ldots), \quad z_n^{(1)}=2\pi ni$$

$$\varphi(z)=\left(1+\frac{2}{\mu z}\right)^{-1}\left[-\kappa_1 z^2-2\left(1+2\kappa_2+\frac{2}{\mu}\right)+\left(\frac{2}{\mu z}-1\right)e^{-z}\right]$$

from which we obtain

$$z_n=2\pi ni+\ln[4\pi^2n^2\kappa_1-2(1+2\kappa_2+2\mu^{-1})]+o(1), \quad n\to\infty$$

The solution of the matrix Riemann equation has the form (1.17) where

$$g_1=(1+2\mu\kappa_0^{-1})^{-1/2}$$

The coefficients A_n are determined by relations (1.18), (1.19), and Δ_n is given by formulae (1.14) where one takes

$$G_n=\mu e_n[-\cos^2\beta_n+\beta_n\sin 2\beta_n+3\kappa_1\beta_n^2-\kappa_2+2e_n\beta_n\cos^2\beta_n(\cos^2\beta_n-\kappa_1\beta_n^2+\kappa_2)],$$
$$e_n=(\sin\beta_n\cos\beta_n+\beta_n)^{-1}$$

The solution of Eq. (2.4) has the form (1.20). On the basis of (2.3) and (1.20) we obtain a formula for the contact stresses

$$\tau_+(x) = -\frac{P_*}{b}\sum_{n=1}^{\infty}\frac{\beta_n A_n}{K_n X_n} e^{\frac{1}{2}\lambda\beta_n}\,\mathrm{sh}\frac{\beta_n x}{b}$$

We will analyse this formula as $x \to -a+0$. To do this we will first study the behaviour of the function $\Phi_1^+(\alpha)$ defined in (1.17) as $\alpha \to 0$ $(0 < \arg\alpha < \pi)$. We have [6]

$$\frac{1}{K^+(\alpha)} = \frac{(-i\alpha)^{-\frac{1}{2}}}{\mu^{\frac{1}{2}}}\left[1 + \frac{\pi}{8i\alpha} + O\left(\frac{1}{\alpha^2}\right)\right], \quad \alpha \to \infty,\ |\arg(-i\alpha)| < \frac{\pi}{2} \tag{2.6}$$

If $\alpha \to \infty$ in the domain $\mathbf{C}^+$ and also $|\alpha - i\beta_n| > \varepsilon$ for any sufficiently small $\varepsilon > 0$ and any $n = 1, 2, \ldots$, then as a consequence of (1.10)

$$\Psi^-(\alpha) = \frac{A^{(0)}}{i\alpha} - \frac{A^{(1)}}{(-i\alpha)^2} + O\left(\frac{1}{\alpha^3}\right), \quad \alpha \to \infty;\ A^{(k)} = \sum_{m=1}^{\infty}\beta_m^k A_m \tag{2.7}$$

We will consider the behaviour of the function $[\mathbf{X}^+(\alpha)]^{-1}$ at infinity. We represent the function $G_0(x)$ in the form

$$G_0(x) = \left(1 + \frac{\mathrm{th}\,x}{\mu x}\right)G_*(x), \quad G_*(x) = 1 + \frac{(\kappa_1 x^2 + \kappa_2)\,\mathrm{th}\,x - x}{(\mathrm{sh}\,x\,\mathrm{ch}\,x + x)[1 + (\mu x)^{-1}\,\mathrm{th}\,x]}$$

and also take into account the asymptotic expansion of the integral

$$\frac{i\alpha}{\pi}\int_1^{\infty}\ln\left(1 + \frac{\mathrm{th}\,x}{\mu x}\right)\frac{dx}{x^2 + (-i\alpha)^2} = \frac{\ln(-i\alpha)}{\pi\mu i\alpha} + \frac{u_0}{i\alpha} + O\left(\frac{1}{\alpha^2}\right)$$

$$\alpha \to \infty,\ |\arg(-i\alpha)| < \pi/2;\ u_0 = \frac{1}{\pi}\int_1^{\infty}\left[\ln\left(1 + \frac{\mathrm{th}\,x}{\mu x}\right) - \frac{1}{\mu x}\right]dx$$

Based in (1.7) we obtain the result

$$\begin{gathered}\frac{1}{\mathbf{X}^+(\alpha)} = 1 + \frac{\ln(-i\alpha)}{\pi\mu i\alpha} + \frac{u_1}{i\alpha} + \sum_{m=2}^{\infty}\frac{1}{(-i\alpha)^m}\sum_{k=0}^{m}c_{nk}^{(0)}\ln^k(-i\alpha) \\ \alpha \to \infty,\ |\arg(-i\alpha)| < \pi/2 \\ u_1 = \frac{1}{\pi}\int_0^1 \ln G_0(x)\,dx + \frac{1}{\pi}\int_1^{\infty}\ln G_*(x)\,dx + u_0\end{gathered} \tag{2.8}$$

which defines the coefficients $c_{mk}^{(0)}$. Substituting the expansions (2.6)–(2.8) into (1.17) we obtain

$$\begin{gathered}-i\alpha\Phi_1^+(\alpha) \sim (-i\alpha)^{-\frac{1}{2}}\sum_{m=0}^{\infty}\frac{1}{(-i\alpha)^m}\sum_{k=0}^{m}c_{mk}\ln^k(-i\alpha) + \omega(\alpha) \\ \alpha \to \infty,\ |\arg(-i\alpha)| < \pi/2 \\ c_{00} = \mu^{-\frac{1}{2}}(g_1 - A^{(0)}),\quad c_{10} = -(\pi/8 + u_1)c_{00} - \mu^{-\frac{1}{2}}A^{(1)},\quad c_{11} = -\pi\mu^{-\frac{3}{2}}\end{gathered} \tag{2.9}$$

The function $\omega(\alpha)$ decreases like $(-i\alpha)^{-1/2}e^{i\alpha\lambda}$ as $\alpha \to \infty$ $(\alpha \in \mathbf{C}^+)$. The numbers c_{mk} $(m \geqslant 2)$ are expressed in terms of the coefficients of expansions (2.6)–(2.8). Bearing in mind the values of the integrals

$$\int_0^\infty \tau^{-\frac{1}{2}+k} e^{i\alpha\tau} d\tau = \frac{\Gamma(\frac{1}{2}+k)}{(-i\alpha)^{\frac{1}{2}+k}} \qquad (k=0,1)$$

$$\int_0^\infty \tau^{\frac{1}{2}} \ln \tau e^{i\alpha\tau} d\tau = \frac{\pi^{\frac{1}{2}}}{2(-i\alpha)^{\frac{3}{2}}}\left[\psi\left(\frac{3}{2}\right) - \ln(-i\alpha)\right], \quad |\arg(-i\alpha)| < \frac{\pi}{2}$$

($\psi(x)$ is the psi function), and also the relation

$$-i\alpha\Phi_1^+(\alpha) = \int_0^\lambda \chi'(\tau) e^{i\alpha\tau} d\tau$$

and formulae (2.9), we obtain a representation of the function $\chi'(t)$ in the neighbourhood of the point $t=0$

$$\chi'(t) \sim \sum_{m=0}^\infty t^{m-\frac{1}{2}} \sum_{k=0}^m D_{mk} \ln^k t + \Omega(t), \quad t \to 0$$

$$D_{00} = (\pi\mu)^{-\frac{1}{2}}(g_1 - A^{(0)}), \quad D_{11} = 2(\pi\mu)^{-\frac{3}{2}}$$

$$D_{10} = -2(\pi/8 + u_1) D_{00} - 2(\pi\mu)^{-\frac{1}{2}} A^{(1)} - 2(\pi\mu)^{-\frac{3}{2}} \psi(\tfrac{3}{2})$$

The remaining coefficients D_{mk} $(m \geqslant 2)$ are computed in terms of the c_{mk}. The function $\Omega(t)$ is infinitely differentiable in the interval $[0, \lambda_*]$ for all $\lambda_* < \lambda$ and has the form

$$\Omega(t) = -\sum_{n=1}^\infty \frac{\beta_n A_n}{K_n X_n} e^{\beta_n t}$$

Using relation (2.3), we obtain the asymptotic expansion

$$\tau_+(x) \sim \frac{P_*}{2b}\{\Omega\left(\frac{x+a}{b}\right) + D_{00}\left(\frac{x+a}{b}\right)^{-\frac{1}{2}} + D_{11}\left(\frac{x+a}{b}\right)^{\frac{1}{2}} \ln\frac{x+a}{b} + \tag{2.10}$$

$$+ D_{10}[(x+a)/b]^{\frac{1}{2}} + \ldots\}, \quad x \to -a+0$$

A similar expansion is obtained in the neighbourhood of the point $x=a$. We define the stress intensity factors

$$K_{\mathrm{II}}(\pm a) = \lim_{x \to \pm a \mp 0} [2\pi(a \mp x)]^{\frac{1}{2}} \tau_+(x)$$

and from the expansion (2.10), using the oddness of the function $\tau_+(x)$, we find

$$K_{\mathrm{II}}(\pm a) = \mp P_* (2\mu b)^{-\frac{1}{2}} (g_1 - A^{(0)})$$

We obtain a formula for the axial force P from the relation [2]

$$P = \int_{-h/2}^{h/2} [q + \sigma_x(a,y)] dy =$$

$$= qh + 2[\partial U/\partial y(a, h/2) - \partial U/\partial y(a,0)] \tag{2.11}$$

Let $U_\alpha(y)$ be the Fourier transform of the function $U(x,y)$. Then, from the solution of problem (2.2) we have

$$\frac{d}{dy} U_\alpha(y) = \frac{\alpha b P_* e^{i\alpha a} \Phi_1^-(\alpha b)}{2(\operatorname{sh} 2\alpha b + 2\alpha b)} \{(y-b)[2\operatorname{ch}\alpha y - (1+\nu)\alpha b \operatorname{sh}\alpha y] +$$

$$+ \alpha^{-1} \operatorname{sh}\alpha(y-b)[2\operatorname{ch}\alpha b - (\nu - 1)\alpha y \operatorname{sh}\alpha b]\}$$

Substituting the last expression into (2.11) and using (2.5), we find that

$$P = qh(1+R)^{-1}(1+E_0E^{-1}R)$$
$$R = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \alpha\Phi_1^-(\alpha)r(\alpha)\,d\alpha \tag{2.12}$$
$$r(\alpha) = (\operatorname{sh}2\alpha+2\alpha)^{-1}\{(\lambda_0-1)[2\operatorname{ch}\lambda_0\alpha-(1+\nu)\alpha\operatorname{sh}\lambda_0\alpha]+$$
$$+\alpha^{-1}\operatorname{sh}(\lambda_0-1)\alpha[2\operatorname{ch}\alpha-\lambda_0(\nu+1)\alpha\operatorname{sh}\alpha]\}, \quad \lambda_0 = h(2b)^{-1}$$

To compute the integral (2.12) we use the relation

$$e^{-i\alpha a}\int_{-a}^{a}\tau_+(x)e^{i\alpha x}\,dx = -\frac{i\alpha b}{2}P_*\Phi_1^-(\alpha b)$$

and obtain

$$R = \frac{1}{\pi}\sum_{n=1}^{\infty}A_n^*R_n, \quad A_n^* = \frac{\beta_nA_n}{K_nX_n}e^{\lambda\beta_n} = O(n^{-1/2}), \quad n\to\infty$$
$$R_n = \int_0^{\infty}\frac{r(\alpha)}{\alpha^2+\beta_n^2}\left[2\alpha(1-e^{-\lambda\beta_n})\cos^2\frac{\lambda\alpha}{2}-\beta_n(1+e^{-\lambda\beta_n})\sin\lambda\alpha\right]d\alpha \tag{2.13}$$

We will improve the convergence of the series (2.13). We have

$$R_n = -\frac{r_0}{\beta_n}+O\left(\frac{1}{\beta_n^2}\right), \quad n\to\infty; \quad r_0 = \int_0^{\infty}\sin\lambda\alpha r(\alpha)\,d\alpha$$
$$A_n^* = \tfrac{1}{2}(\mu\beta_n)^{-1/2}(g_1-A^{(0)})+O(n^{-3/2}\ln n), \quad n\to\infty$$

and arrive at a computationally more convenient relation (where $\zeta(x)$ is the Riemann zeta function)

$$R = -Q\zeta\left(\frac{3}{2}\right)+\sum_{n=1}^{\infty}d_n, \quad d_n = O\left(\frac{\ln n}{n^{5/2}}\right), \quad n\to\infty$$
$$d_n = \pi^{-1}A_n^*R_n+Qn^{-3/2}, \quad Q = \mu^{-1/2}\pi^{-3/2}r_0(q_1-A^{(0)})$$

Note that the coefficients $B_n = e^{\lambda\beta_n}A_n$ can be computed not only with the help of formulae (1.18) and (1.19), but also by the following iterative formulae

$$B_n^{(k)} = \Delta_n\left(\frac{g_1}{\beta_n}-\sum_{m=1}^{\infty}\frac{e^{-\lambda\beta_n}}{\beta_n+\beta_m}B_m^{(k-1)}\right) \quad (k=2,3,\ldots), \quad B_n^{(1)} = \frac{\Delta_ng_1}{\beta_n}$$

where $B_n^{(k)}$ is the kth approximation to the coefficient B_n.

Numerical calculations performed for the problem of the stretching of an infinite strip with an elastic inclusion. Below we give the values of the dimensionless quantities $P^0 = 2(qh)^{-1}P$ (P being the axial force at the end of the stringer) and $P_*^0 = 2(qh)^{-1}P_*$ in the case when $\nu = 0.3$, $\lambda = 2ab^{-1} = 10$, and $\lambda_0 = h(2b)^{-1} = 0.01$ for some values of $k = E_0E^{-1}$

k	0.1	1	2	5	10	100	1000
P^0	1.54	2	2.48	3.60	4,98	13.4	19.1
P_*^0	1.34	0	−1.52	−6.40	−15.0	−187	−1981

Table 1 gives the values of the function $\tau_+^0(x) = -10^4P_*^{-1}\tau_+(x)$ for some values of x and k, corresponding to the same values of ν, λ and λ_0.

TABLE 1

$a^{-1}x$	$k = 0{,}1$	2	10	100
0,1	−0,001	−0,029	−0,030	4,19
0,3	−0,006	−0,067	0,815	21,3
0,5	0,065	1,65	13,0	80,7
0,7	0,901	18,8	96,7	275
0,9	11,1	235	956	1140
0,95	47,4	999	2915	2106

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